ON THE STRUCTURE OF AN INCLINED MAGNETOHYDRODYNAMIC SHOCK WAVE

(O STRUKTURE NAKLONNOI MAGNITOGIDRODINAMICHESKOI UDARNOI VOLNY)

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The problem of the structure of a magnetohydrodynamic shock wave is the problem of finding for a non-ideal gas the solution of the equations of magnetohydrodynamics that assumes at $x = \pm\infty$ the values which satisfy the known conservation laws for transition through a sharp discontinuity surface in an ideal gas. In the study of the structure of a shock wave it is assumed that in a coordinate system in which the wave is at rest the motion within the zone representing the shock wave is the steady one-dimensional motion of a non-ideal gas. In this formulation the problem of the structure of a parallel shock wave was considered in [1]. Some general questions on the structure of an inclined shock wave with all dissipative coefficients included were considered in [2]. The present paper is an investigation of the flow within the shock-wave zone in the case when dissipation of energy in the wave arises from consideration of magnetic viscosity and the second kinematic viscosity. With an analogous formulation the structure of a certain special kind of shock wave was considered in [3].

The equations of steady one-dimensional flow of a perfect gas, describing the problem of shock-wave structure when only the magnetic viscosity and second viscosity are different from zero, have the form

$$\nu_m \frac{dH}{dx} = uH - vH_n + cE, \qquad \mu \frac{du}{dx} = p + \rho u^2 + \frac{1}{8\pi} H^2 - J_1$$

$$\rho uv - \frac{1}{4\pi} H_n H = J_2, \qquad \rho u = M, \qquad H_n = \text{const} \tag{1}$$

$$\rho u \left[\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2)\right] - \frac{cEH}{4\pi} = U$$

Here H_n, H, u, v are respectively the components of the magnetic field and velocity along the x- and y-axes, E the z-component of the electric field, c the speed of sound, J_1 and J_2 the x- and y-components of momentum flux, U the energy flux, and M the mass flux. Equations (1) are written in a system of coordinates in which the flow may be considered plane. At $x = \pm\infty$ the flow representing the structure of the shock wave should approach a uniform flow with parameters satisfying the conservation laws. Thus in Equations (1) we have $du/dx = dH/dx = 0$, and the constants U, J_1, J_2, E, H_n, M can be determined from the values of the parameters ahead of the shock wave. Without loss of generality it is possible to take $J_2 = 0$. We introduce dimensionless variables according to the following equations (where we refer all quantities to the parameters ahead of the shock wave):

$$u = u_0\tau, \quad v = u_0 q, \quad p = \rho_0 u_0^2\theta, \qquad H = \sqrt{4\pi\rho_0 u_0^2}\,h \tag{2}$$

In the new variables Equations (1) can be written in the following form:

$$\frac{\nu_m}{u_0}\frac{dh}{dx} = h(\tau - h_n^2) - e, \qquad \frac{\mu}{\rho_0 u_0}\frac{d\tau}{dx} = \theta + \tau + \tfrac{1}{2}h^2 - P$$

$$q - h_n h = 0, \qquad k\theta\tau + \tfrac{1}{2}\tau^2 + \tfrac{1}{2}h_n^2 h^2 + eh = \varepsilon \tag{3}$$

$$\left(k = \frac{\gamma}{\gamma - 1},\ h_n = \frac{H_n}{\sqrt{4\pi\rho_0 u_0^2}},\ e = -\frac{cE}{\sqrt{4\pi\rho_0 u_0^2}},\ P = \frac{J_1}{\rho_0 u_0^2},\ \varepsilon = \frac{U}{\rho_0 u_0^3}\right)$$

In the first and last of Equations (3) q has been eliminated by means of the third equation. From the condition that at $x = -\infty$ the derivatives dh/dx and $d\tau/dx$ vanish and $\tau = 1$ it follows that

$$e = h_0(1 - h_n^2), \qquad P = 1 + \theta_0 + \tfrac{1}{2}h_0^2, \qquad \varepsilon = k\theta_0 + \tfrac{1}{2} + h_0^2\left(1 - \tfrac{1}{2}h_n^2\right)$$

Henceforth we will always assume that $e > 0$, that is, we will assume that the sign of h_0 is the same as the sign of $(1 - h_n^2)$. From the last relation we obtain

$$\frac{1}{2}h_0^2 = \frac{k(P-1) - (\varepsilon - 1/2)}{h_n^2 + k - 2}, \qquad \theta_0 = \frac{(\varepsilon - 1/2) - (2 - h_n^2)(P - 1)}{h_n^2 + k - 2} \tag{4}$$

For simplicity we will henceforth consider the case $\gamma < 2$. Then

$$k > 2, \qquad h_0^2 + k - 2 > 0$$

Since $h_0^2 \geqslant 0$ and $\theta_0 \geqslant 0$, it is possible to obtain from (4) an inequality bounding the range of possible values of the constant ϵ

$$(2 - h_n^2)(P - 1) \leqslant \varepsilon - \tfrac{1}{2} \leqslant k(P - 1) \tag{5}$$

After this preliminary remark we turn to an investigation of the isoclines $d\tau/dx = 0$. Eliminating θ from the second and fourth of Equations (3), we obtain their equation

$$h^2(k\tau - h_n^2) - 2eh + (2k - 1)\tau^2 - 2kP\tau + 2\varepsilon = 0 \tag{6}$$

It is easy to verify that the point $\tau = 1$, $h = h_0$, satisfying conditions ahead of the wave, satisfies Equation (6).

We solve Equation (6) for h:

$$h_{1,2} = \frac{e}{k\tau - h_n^2} \pm \frac{\sqrt{e^2 - (k\tau - h_n^2)[(2k - 1)\tau^2 - 2kP\tau + 2\varepsilon]}}{k\tau - h_n^2} \tag{7}$$

Thus real points of the isocline $d\tau/dx = 0$ lie on both sides of the hyperbola $h = e/(k\tau - h_n^2)$. Minima and maxima of the isocline lie on this hyperbola at points τ where the discriminant of Equation (6) vanishes. It is evident immediately from Equation (7) that the isocline $d\tau/dx = 0$ has the asymptote $\tau = h_n^2/k$.

Introducing the quantity α in place of ϵ by means of the equation

$$2\varepsilon - 1 = 2k(P - 1) - 2\alpha(h_n^2 + k - 2)(P - 1)$$

and using the expression for e, we put the discriminant of Equation (6) in the form

$$D(\tau) = 2\alpha(P - 1)[(1 - kh_n^2) + k(h_n^2 + k - 2)\tau] - \\ - (k\tau - h_n^2)[(2k - 1)\tau^2 - 2kP\tau + 2k(P - 1) + 1]$$

The quantity α varies within the limits

$$0 \leqslant \alpha = \frac{{}^1\!/_2\, h_0^2}{{}^1\!/_2\, h_0^2 + \theta_0} \leqslant 1$$

For $\alpha = 0$ the equation $D(\tau) = 0$ has three real roots

$$\tau = \frac{h_n^2}{k}, \qquad \tau = 1, \qquad \tau = \tau_1 \equiv \frac{2k}{2k - 1}(P - 1) + \frac{1}{2k - 1}$$

As α increases, the root of $D(\tau) = 0$, that is, the point of intersection of the straight line

$$D(\tau) - D(\tau)\big|_{\alpha=0} = 2\alpha k(h_n^2 + k - 2)(P-1)(\tau - \tau_*)$$
$$\left(\tau_* = \frac{h_n^2 - 1/k}{h^2 + k - 2}\right)$$

with the cubic parabola

$$-D(\tau)\big|_{\alpha=0} = (2k-1)(k\tau - h_n^2)(\tau - \tau_1)(\tau - 1)$$

increases monotonically. This permits a qualitative investigation of the question of the number and distribution of roots of the discriminant.

The results of this investigation can be stated in the following form. In the plane of the variables $P - 1 = 1/2\ h_0^2 + \theta_0$ and h_n^2 a curve can be drawn for $\theta_0 = 0$ ($\alpha = 1$) separating the region of existence of three roots of the discriminant from the region where only one root exists. It has the form shown in Fig.1 (the curve $ABCD$). The curve ECF, whose equation is $\tau_1 = \tau_*$, is tangent to the curve $ABCD$ at the point C. For points lying to the left of the curve $ABCF$ the discriminant has always (for arbitrary α) three roots. For points lying to the right of the curve $ABCD$ the discriminant has three roots for small values of α and one root for large values of α. For the remaining points in the $P - 1$, h_n^2-plane the discriminant has three roots for small and large values of α, and one root for intermediate values.

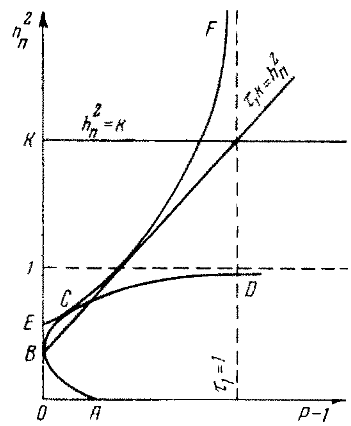

Fig. 1.

We introduce also in the $P - 1$, h_n^2-plane the straight lines

$$\frac{h_n^2}{k} = 1, \qquad \tau_1 = 1, \qquad \tau_1 = \frac{h_n^2}{k} \qquad \left(\tau_1 \equiv \frac{2k}{2k-1}(P-1) + \frac{1}{2k-1}\right)$$

In the case when $h_0 = 0$, the values 1, τ_1 and h_n^2/k are roots of $D(\tau)$. The straight line $\tau_1 = h_n/k$ passes through point B and is tangent to the curve ECF at $h_n^2 = 1$. The straight line $\tau_1 = 1$ is the asymptote of the curve ECF.

It can be shown that:

a) for points lying simultaneously below the straight lines $\tau_1 = h_n^2/k$ and $h_n^2/k = 1$ in those cases when the number of roots is three and does not change as a varies from zero to the value under consideration, two roots of the discriminant are greater than h_n^2/k and one is smaller:

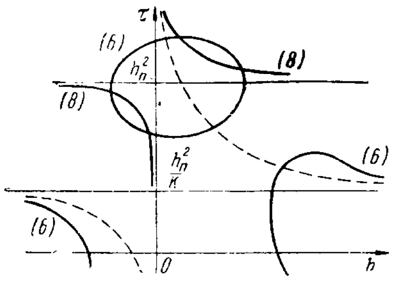

Fig. 2.

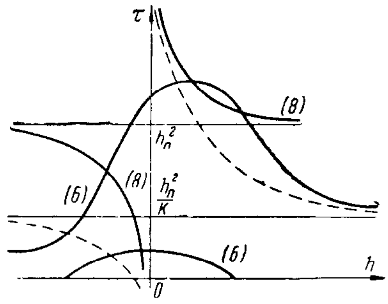

Fig. 3.

b) in all other cases when there are three roots, one root is greater than h_n^2/k and the other two are smaller.

These properties of the discriminant permit the isocline $d\tau/dx = 0$ in the h, τ-plane to be constructed qualitatively according to Equation (7). If there are three roots, then in case (a) the isocline will have the form of curve (6) in Fig. 2. The dotted line indicates the hyperbola $h = e/(k\tau - h_n^2)$. The point $(1, h_0)$, corresponding to the initial condition, always lies on the closed branch. In case (b) the isocline will have the form of curve (6) in Fig. 3.

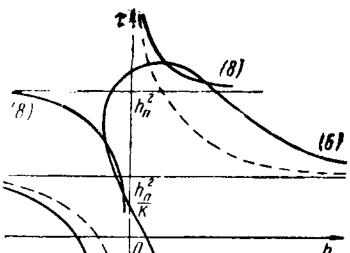

Fig. 4.

We note that in this case the initial point may lie either above or below the asymptote $\tau = h_n^2/k$, and the point of intersection of the isocline with the asymptote may be to the right or left of the τ-axis. However, if the discriminant has only one root the isocline has the form of curve (6) in Fig. 4. The isocline $dh/dx = 0$ in the h,τ-plane is the hyperbola

$$h(\tau - h_n^2) = e \qquad (8)$$

with asymptotes $h = 0$ and $\tau = h_n^2$; it is indicated by the numeral (8) in Figs. 2, 3 and 4.

From the determination of h_n it follows that for points above the line $\tau = h_n^2$ the gas speed is greater than the Alfven speed $a_A = H_n/\sqrt{(4\pi\rho_0)}$, and for points below the line $\tau = h_n^2$ the speed is less than that.

Since for $x = \pm\infty$ the flow corresponding to the problem of the shock-wave structure should tend to a uniform stream ($dh/dx = d\tau/dx = 0$), points corresponding to conditions behind and ahead of the shock are the points of intersection of the isoclines (6) and (8). These points of intersection are the singular points of the equation

$$\frac{d\tau}{dh} = \frac{h^2(k\tau - h_n^2) - 2eh + (2k-1)\tau^2 - 2kP\tau + 2\varepsilon}{2k\tau\,[h(\tau - h_n^2) - e]} \tag{9}$$

equivalent to the two differential equations (3). Solutions of the problem of shock-wave structure correspond to the integral curves of Equation (9) joining the singular points lying in the region $\tau > 0$. In order to investigate the field of integral curves it is first of all necessary to study the character of these singular points.

It is shown in [2] that the character of a singular point of Equation (9) in the h, τ-plane depends on the value of the velocity corresponding to the given singular point. Thus the following singular points are possible (enumerated in order of decreasing velocity):

Point 1 -- a node, from which integral curves issue as x increases. At this point the inequality $a_+ < u$ is satisfied, or

$$\frac{1}{2}\left\{\sqrt{\gamma\theta\tau + h^2\tau + 2h_n\tau\sqrt{\gamma\theta}} + \sqrt{\gamma\theta\tau + h^2\tau - 2h_n\tau\sqrt{\gamma\theta}}\right\} \equiv \frac{a_+}{u_0} < \tau$$

Point 2 -- a saddle point. At this point the following inequality is satisfied:

$$a_A < u < a_+ \qquad \text{or} \qquad h_n^2\tau^{-\frac{1}{2}}\tau < \frac{a_+}{u_0}$$

Point 3 -- a saddle point. At this point the following inequality is satisfied:

$$a_- < u < a_A$$

or

$$\frac{1}{2}\left\{\sqrt{\gamma\theta\tau + h^2\tau + 2h_n\tau\sqrt{\gamma\theta}} - \sqrt{\gamma\theta\tau + h^2\tau - 2h_n\tau\sqrt{\gamma\theta}}\right\} \equiv \frac{a_-}{u_0} < \tau < h_n^2\tau^{-1/2}$$

Point 4 -- a node, into which integral curves enter. At this point the following inequality is satisfied:

$$u < a, \qquad \text{or} \qquad \tau < \frac{a_-}{u_0}$$

Here a_+ and a_- are the fast and slow magneto-acoustic speeds.

If all four singular points occur, two of them lie above the line $\tau = h_n^2$ and the other two below it. If there are only two singular points they lie on one side of this line. The transition $1 \to 2$ is called the fast magnetohydrodynamic shock wave, the transition $3 \to 4$ is called the slow magnetohydrodynamic shock wave, and all other transitions are intermediate shock waves. If two singular points coalesce into one (the isoclines (6) and (8) become tangent) the stream velocity at this point is equal to one of the magneto-acoustic speeds and the shock wave becomes a weak perturbation.

It can be shown that if the curve (6) has the form indicated in Figs. 2 and 4 all singular points lie on one branch of curve (6).

The investigation of the isoclines (6) and (8) that has been carried out permits qualitative construction of the field of integral curves of the system of differential equations (3). If the isoclines (6) and (8) have the form shown in Fig. 3 the corresponding field of integral curves has, depending on the value of the ratio of dissipation coefficients $\mu/\rho_0\nu_m$, the form shown in Figs. 5, 6 and 7. In all other cases, if there are four singular points, the character of the integral curves does not change*. If there are two singular points (either pair) the behavior of

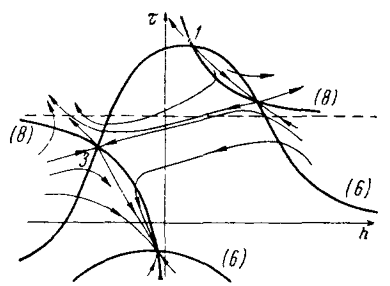

Fig. 5.

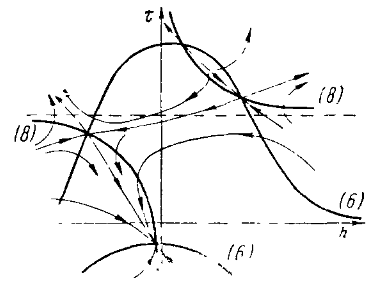

Fig. 6.

the integral curves in the vicinity of these points remains the same as in the presence of four singular points.

The picture of the integral curves shown in Fig. 5 corresponds to the case when the ratio $\mu/\rho_0\nu_m$ is small. In this case points 1 and 2, 3 and

* In the case of the disposition of isoclines shown in Fig. 2, the behavior of the integral curves of Equation (9) was studied in the thesis of A.N. Voinov (Moscow State University, Faculty of Mechanics and Mathematics, 1960).

4 are connected together in pairs by unique integral curves representing, respectively, the structure of the fast and slow shock waves. The points of the first pair and points of the second pair are not connected by integral curves, which corresponds to the absence of the structure of intermediate shock waves. If $\mu/\rho_0\nu_m \to 0$ the integral curves representing the structure of the fast and slow shock wave tend to coincidence with the corresponding segments of the isocline $dh/dx = 0$.

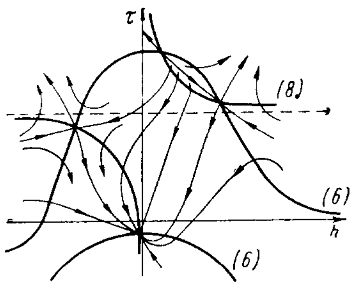

Fig. 7.

Figure 6 corresponds to a unique value of the ratio $\mu/\rho_0\nu_m = (\mu/\rho_0\nu_m)_*$ for which the integral curve leaving point 2 enters point 3. This value of $\mu/\rho_0\nu_m$ separates the cases shown in Figs. 5 and 7. Here the integral curves connect points $1 \to 2$ (fast wave), $3 \to 4$ (slow wave) and $2 \to 3$ (intermediate wave), the integral curves connecting these singular points being unique. The more complicated transitions $1 \to 2 \to 3$, $2 \to 3 \to 4$, $1 \to 2 \to 3 \to 4$ are also possible.

For large values of $\mu/\rho_0\nu_m$ there exists the case shown in Fig. 7, where the integral curves connect the following pairs of singular points: $1 \to 2$, $3 \to 4$, $1 \to 3$, $1 \to 4$, $2 \to 4$, all these pairs of points being connected by a single integral curve except for the pair $1 \to 4$, which is connected by an infinite number of integral curves. There exist the following compound transitions: $1 \to 2 \to 4$, $1 \to 3 \to 4$.

If $\mu/\rho_0\nu_m \to \infty$ the integral curves tend to coincidence with the isocline $d\tau/dx = 0$. However, in this case, when the motion is supersonic at the initial point and subsonic at the final point, a gasdynamic shock wave arises within the structure of the shock wave, with $H =$ const. We note that if such a shock wave exists the structure of the fast magnetohydrodynamic shock wave is terminated by the gasdynamic shock, and the structure of the slow shock wave begins with the gasdynamic shock. The structure of the intermediate shock waves may also include a gasdynamic shock, which appears at the beginning of the $1 \to 3$ shock, at the end of the $2 \to 4$ shock and at an arbitrary point in the structure of the $1 \to 4$ shock.

Thus in the formulation considered the fast and slow wave possess structure for arbitrary ratios of the dissipative coefficients.

In cases when there are four singular points the intermediate shock waves may possess structure.

The transition 2 → 3 is possible only for

$$\frac{\mu}{\rho_0 \nu_m} = \left(\frac{\mu}{\rho_0 \nu_m}\right)_*$$

The transitions 1 → 3 and 2 → 4 exist and are unique for

$$\frac{\mu}{\rho_0 \nu_m} > \left(\frac{\mu}{\rho_0 \nu_m}\right)_*$$

The transition 1 → 4 is possible for

$$\frac{\mu}{\rho_0 \nu_m} > \left(\frac{\mu}{\rho_0 \nu_m}\right)_*$$

and may proceed by an infinite number of integral curves.

It is interesting to note that in the given formulation of the problem of the structure of a shock wave the evolutionary shock wave in the sense of [4] differs from the non-evolutionary in that it alone possesses structure for arbitrary ratios between the dissipative coefficients.

Note. The statement of Germain [2] that in the formulation under consideration the slow shock wave does not always possess structure is false. It is based on the supposition that the point of intersection of the upper branch of the isocline $dr/dx = 0$ with the straight line $h = h_0$ passing through point 3 may lie below the h-axis. However, this point is connected with point 3 by transition through an ordinary gasdynamic shock, and must lie in the region $r > 0$ if the pressure at point 3 is positive.

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